Symplectic structures on quadratic Lie algebras

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Abstract

We study quadratic Lie algebras over a field $\mathbb{K}$ of null characteristic which admit, at the same time, a symplectic structure. We see that if $\mathbb{K}$ is algebraically closed every such Lie algebra may be constructed as the T*-extension of a nilpotent algebra admitting an invertible derivation and also as the double extension of another quadratic symplectic Lie algebra by the one-dimensional Lie algebra. Finally, we prove that every symplectic quadratic Lie algebra is a special symplectic Manin algebra and we give an inductive classification in terms of symplectic quadratic double extensions.

Introduction

Lie groups which admit a bi-invariant pseudo-Riemannian metric and also a left-invariant symplectic form are nilpotent Lie groups and their geometry (and, consequently, that of their associated homogeneous spaces) is very rich. In particular, they carry two left-invariant affine structures: one defined by the symplectic form (which is well-known) and another which is compatible with a left-invariant pseudo-Riemannian metric. Moreover, if the symplectic form is viewed as a solution r of the classical Yang-Baxter equation, then the Poisson-Lie tensor $\pi = r^+ - r^-$ and the geometry of the double Lie groups $D(r)$ can be nicely described [7]. As we will see below, a great number of such Lie groups may be constructed and, hence, a large class of symplectic symmetric spaces is found.

The Lie algebra of one of those Lie groups turns to be a *quadratic symplectic Lie algebra*, this is to say, a Lie algebra endowed with both an invariant non-degenerate symmetric bilinear form and a non-degenerate 2-cocycle of its scalar cohomology. In this paper we study the structure of these nilpotent Lie algebras over a field $\mathbb{K}$ of characteristic zero and give some results which provide a method for their inductive classification. The main tools used for our purposes are T*-extensions and quadratic double extensions.

The notion of T*-extension of a Lie algebra was introduced by M. Bordemann in [4], where it is proved that every solvable quadratic Lie algebra over an algebraically closed field of characteristic 0 is either a T*-extension or a non-degenerate ideal of codimension 1 in a T*-extension of some Lie algebra. In general, a T*-extension of a Lie algebra need not admit a non-degenerate scalar 2-cocycle, even if the extended algebra is nilpotent. Thus,

additional properties should be imposed to a Lie algebra $\mathfrak{g}$ to ensure that a T*-extension $T^*_\theta\mathfrak{g}$ might be furnished with a symplectic structure. This is done in section 2, where we show that if $\mathbb{K}$ is algebraically closed, then every quadratic symplectic Lie algebra is a T*-extension of a Lie algebra which admit an invertible derivation and we give necessary and sufficient conditions on $\mathfrak{g}$ and on the cocycle θ used in the construction of the T*-extension $T^*_\theta\mathfrak{g}$ to assure that the extended algebra admits an skew-symmetrical derivation and, hence, a symplectic structure. We use these results to give a complete classification of complex quadratic Lie algebras of dimension less than or equal to 8 which admit a symplectic structure.

Every n-dimensional solvable quadratic Lie algebra may be obtained from a quadratic Lie algebra of dimension $n-2$ by a central extension by a one-dimensional algebra and a semi-direct product by another one-dimensional algebra. This method of construction, known as quadratic double extension, was introduced and developped for the first time in [12], [13]. Later, Dardié and Medina addapted and generalised this method to study certain symplectic Lie groups [6]. In A. Aubert's unpublished thesis [3], these methods of (generalised) symplectic double extension are used to study quadratic symplectic Lie algebras. A restatement of some of her results using quadratic double extension instead of symplectic double extension is given in section 3 since it will be useful in the last section of the paper, where we study Manin algebras admitting an adapted symplectic form, which we will call *special symplectic Manin algebras*. This study is motivated by the fact that every symplectic quadratic Lie algebra over an algebraically closed field is actually a special symplectic Manin algebra. We give a description of these Lie algebras in term of quadratic double extension and obtain that, if the field is algebraically closed, every such special symplectic Manin algebra may be obtained from a two-dimensional symplectic Manin algebra by a sequence of quadratic double extensions by the one-dimensional Lie algebra where the algebra obtained in each step is also a special symplectic Manin algebra.

1 Definitions and preliminary results

All through the paper $\mathbb{K}$ will denote a commutative field of characteristic 0. We begin with the following:

Definition 1.1 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$.

i) We say that $(\mathfrak{g}, B)$ is a *quadratic Lie algebra* if B is a non-degenerate symmetric bilinear form on $\mathfrak{g}$ such that $B([x,y],z) = B(x,[y,z])$ for all $x,y,z \in \mathfrak{g}$. In that case, we will say that B is an *invariant scalar product* on $\mathfrak{g}$.

   A quadratic Lie algebra $(\mathfrak{g}, B)$ is said to be *reducible* (or B-reducible) if it admits an ideal $\mathfrak{I}$ such that the restriction of B to $\mathfrak{I} \times \mathfrak{I}$ is non-degenerate. We will say that $(\mathfrak{g}, B)$ is *irreducible* otherwise.

ii) We say that $(\mathfrak{g}, \omega)$ is a *symplectic Lie algebra* if ω is a non-degenerate skew-symmetric bilinear form on $\mathfrak{g}$ and $\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y) = 0$ holds for all $x, y, z \in \mathfrak{g}$, this is to say, ω is a non-degenerate 2-cocycle for the scalar cohomology of $\mathfrak{g}$. Note that in such case, $\mathfrak{g}$ must be even-dimensional. We will then call ω a *symplectic structure* on $\mathfrak{g}$.

iii) We will say that $(\mathfrak{g}, B, \omega)$ is a *quadratic symplectic Lie algebra* if $(\mathfrak{g}, B)$ is quadratic and $(\mathfrak{g}, \omega)$ is symplectic.

Lemma 1.1 *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra over $\mathbb{K}$. A symplectic structure ω may be defined on $\mathfrak{g}$ if and only if there exists a skew-symmetric invertible derivation D of $(\mathfrak{g}, B)$.*

Proof. It is a straightforward calculation considering $\omega(x,y) = B(Dx, y)$ for all $x, y \in \mathfrak{g}$. □

From now on, given a quadratic Lie algebra $(\mathfrak{g}, B)$, we will denote by $\mathrm{Der}_a(\mathfrak{g}, B)$ the Lie algebra of skew-symmetric derivations of $(\mathfrak{g}, B)$.

Remark 1

1. Since the Lie algebra $\mathrm{Der}_a(\mathfrak{g}, B)$ is algebraic, the existence of an invertible skew-symmetric derivation of $(\mathfrak{g}, B)$ is equivalent to the existence of an invertible semisimple skew-symmetric derivation of $(\mathfrak{g}, B)$.

2. Clearly, every quadratic Lie algebra admitting a symplectic structure (and hence an invertible derivation) must be nilpotent [11].

3. It should be noticed that under the assumptions of the lemma above, the skew-symmetric derivation D of $(\mathfrak{g}, B)$ is also skew-symmetric with respect to the symplectic form ω since for all $x, y \in \mathfrak{g}$ we get

$$\omega(Dx, y) = B(D^2x, y) = -B(Dx, Dy) = -\omega(x, Dy).$$

   One might thus think that every symplectic Lie algebra $(\mathfrak{g}, \omega)$ admitting an invertible derivation which is skew-symmetric for ω carries a quadratic structure; but this is not the case. For example, the four dimensional Lie algebra $\mathfrak{g} = \mathbb{R}\text{-span}\{x_1, x_2, x_3, x_4\}$ defined by the only non-trivial bracket $[x_1, x_2] = x_3$ does not admit any quadratic structure since $[\mathfrak{g}, \mathfrak{g}]$ is one dimensional while the centre of $\mathfrak{g}$ has dimension two. However, the skew-symmetric bilinear form on $\mathfrak{g}$ given by $\omega(x_1, x_4) = \omega(x_2, x_3) = 1$ provides a symplectic structure on $\mathfrak{g}$ and the linear endomorphism of $\mathfrak{g}$ given by $D(x_1) = 2x_1$, $D(x_2) = -x_2$, $D(x_3) = x_3$, $D(x_4) = -2x_4$, is a skew-symmetric derivation of $(\mathfrak{g}, \omega)$.

The following example shows that, starting from an arbitrary Lie algebra, one can construct infinitely many quadratic symplectic Lie algebras.

Example 1 Let $\mathfrak{g}$ be a Lie algebra and $n \in \mathbb{N},\ n > 1$. If we consider the non-unitary associative algebra $\mathcal{A}_n = t\mathbb{K}[t]/t^n\mathbb{K}[t]$, Then the vector space $\mathcal{L}_n = \mathfrak{g}\otimes\mathcal{A}_n$ with the bracket

$$[x \otimes \bar{t^p}, y \otimes \bar{t^q}] = [x, y]_{\mathfrak{g}} \otimes \overline{t^{p+q}}, \quad x, y \in \mathfrak{g},\ p, q \in \mathbb{N} \setminus \{0\},$$

is a nilpotent Lie algebra. The endomorphism D of $\mathcal{L}_n$ defined by $D(x \otimes \bar{t^p}) = p(x \otimes \bar{t^p})$, for all $x \in \mathfrak{g},\ p \in \{1, \ldots, n-1\}$, is an invertible derivation of $\mathcal{L}_n$.

Now, the vector space $\tilde{\mathcal{L}_n} = \mathcal{L}_n \oplus (\mathcal{L}_n)^*$ with the bracket defined by

$$[X + f, Y + g] = [X, Y]_{\mathcal{L}_n} - g \circ \mathrm{ad}_{\mathcal{L}_n}(X) + f \circ \mathrm{ad}_{\mathcal{L}_n}(Y),$$

for $X, Y \in \mathcal{L}_n,\ f, g \in (\mathcal{L}_n)^*$, and the bilinear form $B(X + f, Y + g) = f(Y) + g(X)$, is a quadratic Lie algebra. Further an invertible skew-symmetric derivation of $\tilde{\mathcal{L}_n}$ may be defined by $\tilde{D}(X+f) = D(X)+D^*(f), \forall X \in \mathfrak{g},\ f \in \mathfrak{g}^*$, where $D^*(f) = -f \circ D$, and hence the quadratic algebra $(\tilde{\mathcal{L}_n}, B)$ admits a symplectic structure.

It is well-known that if $(\mathfrak{g}, \omega)$ is a symplectic Lie algebra then the product $x.y$ on $\mathfrak{g}$ defined by $\omega(x.y, z) = -\omega(y, [x, z])$ induces on every Lie group G with Lie algebra $\mathfrak{g}$ a flat and torsion-free invariant connection by the formula $\nabla^\omega_x y = x.y$, for $x, y \in \mathfrak{g}$. Further, in [15] it is proved that if $\mathfrak{g}$ also admits a quadratic structure then one may define a left-invariant pseudo-Riemannian metric $\langle ., . \rangle$ on G whose Levi-Civita connection is precisely the flat connection ∇^ω. A slight generalisation of such situation may be done since the result remains valid for non-symplectic Lie algebras which admit a (not necessarily skew-symmetric) invertible derivation. In order to prove this, we recall the following well-known result:

Lemma 1.2 *If $\mathfrak{g}$ is a Lie algebra and D is an invertible derivation of $\mathfrak{g}$ then the product $x.y = D^{-1}[x, Dy]$ for $x, y \in \mathfrak{g}$, defines a Lie-admissible complete left-symmetric structure on $\mathfrak{g}$. Further, D is also a derivation of the left-symmetric product.*

Proposition 1.3 *Let $\mathfrak{g}$ be a Lie algebra, D an invertible derivation of $\mathfrak{g}$ and let $(\mathfrak{g}, .)$ be the left-symmetric structure defined by D. The algebra $\mathfrak{g}$ admits a quadratic structure if and only if there exists a non-degenerate symmetric bilinear form $\langle ., . \rangle$ on $\mathfrak{g}$ such that its Levi-Civita connection is given by $\nabla_x y = x.y$, for all $x, y \in \mathfrak{g}$.*

Proof. The proof is almost immediate if we define $\langle ., . \rangle$ or, conversely, the invariant scalar product B according to $\langle x, y \rangle = B(Dx, Dy)$, for all $x, y \in \mathfrak{g}$. □

The following corollary is the obvious geometric analog of the proposition above.

Corollary 1.4 *Let G be a Lie group and suppose that its Lie algebra $\mathfrak{g}$ admits an invertible derivation D. The following are equivalent:*

(i) G admits a bi-invariant pseudo-Riemannian metric.

(ii) G admits a left-invariant flat pseudo-Riemannian metric whose Levi-Civita connection is given by left-translation of the left-symmetric product induced by D on $\mathfrak{g}$.

Remark 2 Quadratic Lie algebras admitting an invertible skew-symmetric derivation provide nice examples of solutions of both classical and modified Yang-Baxter equations. We recall that for a quadratic Lie algebra $(\mathfrak{g}, B)$ a linear endomorphism R of $\mathfrak{g}$ which is skew-symmetric with respecto to B is said to satisfy the classical Yang-Baxter equation if $[Rx, Ry] - R[Rx, y] - R[x, Ry] = 0$ holds for every $x, y \in \mathfrak{g}$. If, instead, for every $x, y \in \mathfrak{g}$ the mapping R verifies $[Rx, Ry] - R[Rx, y] - R[x, Ry] + [x, y] = 0$ one says that R is a solution of the modified Yang-Baxter equation. A straightforward calculation shows that if D is an invertible skew-symmetric derivation of $(\mathfrak{g}, B)$ then $R = D^{-1}$ is a solution of the classical Yang-Baxter equation. Further, one easily sees that in the complex case a quadratic Lie algebra with an invertible skew-symmetric derivation defines naturally a Manin triple and, as a consequence, a solution of the modified Yang-Baxter equation [1], [5], [14].

Notation In all the sections below, the symbol $\oplus$ will be frequently used. In principle, unless other thing is stated, it will only denote direct sum of vector spaces.

2 Quadratic Lie algebras with symplectic structures and the notion of T^*-extension

In [4], M. Bordemann defines the following notion:

Definition 2.1 [4] Let $\mathfrak{a}$ be a Lie algebra over a commutative field and consider its dual space $\mathfrak{a}^*$. Consider a 2-cocycle $\theta : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}^*$ and define on the vector space $T^*_\theta \mathfrak{a} = \mathfrak{a} \oplus \mathfrak{a}^*$ the following bracket:

$$[x + f, y + g] = [x, y]_{\mathfrak{a}} + \theta(x, y) + f \circ \mathrm{ad}_{\mathfrak{a}}(y) - g \circ \mathrm{ad}_{\mathfrak{a}}(x) , \quad x, y \in \mathfrak{a},\ f, g \in \mathfrak{a}^*.$$

The pair $(T^*_\theta \mathfrak{a}, [.,.])$ is then a Lie algebra called the *T^*-extension of the Lie algebra $\mathfrak{a}$ by means of θ.*

If, further, the 2-cocycle θ verifies the *cyclic* condition $\theta(x, y)(z) = \theta(y, z)(x)$ for all $x, y, z \in \mathfrak{a}$, then the symmetric bilinear form B on $T^*_\theta \mathfrak{a}$ given by $B(x+f, y+g) = f(x)+g(y)$ for $x, y \in \mathfrak{a},\ f, g \in \mathfrak{a}^*$, defines a quadratic structure on $T^*_\theta \mathfrak{a}$.

Note that the condition of cyclicity for θ proves that θ defines a 3-cocycle of the scalar cohomology. Actually, the natural mapping between the set of cyclic 2-cocyles and $Z^3(\mathfrak{a}, \mathbb{K})$ given by $\tilde{\theta}(x,y,z) = \theta(x,y)(z)$ for all x, y, z is an isomorphism.

The case $\theta = 0$ corresponds to the double extension of the null algebra by the algebra $\mathfrak{a}$ [13]. This algebra, which we will call the *trivial T^*-extension of* $\mathfrak{a}$, is isometrically isomorphic to $T^*_\theta\mathfrak{a}$ if $\tilde{\theta}$ is a coboundary [4].

Since we will mainly deal with even-dimensional Lie algebras the following particular case of [4, Cor. 3.3] will be sufficient for our purposes:

Proposition 2.1 *Suppose that $\mathbb{K}$ is algebraically closed and let $(\mathfrak{g}, B')$ be a quadratic even-dimensional Lie algebra over $\mathbb{K}$. If $\mathfrak{g}$ is solvable then $(\mathfrak{g}, B')$ is isometrically isomorphic to a quadratic T^*-extension $(T^*_\theta\mathfrak{a}, B)$, where $\mathfrak{a}$ is isomorphic to the quotient algebra of $\mathfrak{g}$ by a completely isotropic ideal.*

From the proposition above, one obviously gets that every Lie algebra admitting both a quadratic and a symplectic structure is isometrically isomorphic to some T^*-extension. However it is clear that not every T^*-extension of a nilponent Lie algebra should admit a symplectic structure. Our main interest is to give necessary and sufficient conditions for a quadratic Lie algebra endowed with a symplectic structure to be a T^*-extension of some subalgebra. This is done in the two following propositions.

Proposition 2.2 *Let $\mathfrak{a}$ be a Lie algebra admitting an invertible derivation D. Consider a cyclic 2-cocycle $\theta \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ and define*

$$\Theta(x,y,z) = \theta(Dx,y)z + \theta(Dy,z)x + \theta(Dz,x)y\,, \quad x,y,z \in \mathfrak{a}.$$

If Θ is a 3-coboundary for the scalar cohomology of $\mathfrak{a}$ then the quadratic Lie algebra $T^*_\theta\mathfrak{a}$ admits a symplectic structure.

Proof. Let B be the quadratic form on $T^*_\theta\mathfrak{a}$ defined in Definition 2.1. By Lemma 1.1 it suffices to prove the existence of an invertible skew-symmetric derivation of $(T^*_\theta\mathfrak{a}, B)$.

Let us consider $F : \mathfrak{a} \wedge \mathfrak{a} \to \mathbb{K}$ be such that $\Theta = dF$ and let $H : \mathfrak{a} \to \mathfrak{a}^*$ be the mapping defined by $B(Hx, y) = F(x, y)$ for all $x, y \in \mathfrak{a}$. Now, define $\overline{D} : T^*_\omega\mathfrak{a} \to T^*_\omega\mathfrak{a}$ by $\overline{D}(x + f) = Dx - Hx - f \circ D$ for all $x \in \mathfrak{a}, f \in \mathfrak{a}^*$. It is straightforward to see that $\overline{D}$ is invertible, since D is so, and a direct calculation shows that $\overline{D}$ is also a skew-symmetric with respect to B.

Further, since D is a derivation of $\mathfrak{a}$, we get

$$\begin{aligned}&[\overline{D}(x+f), y+g] + [x+f, \overline{D}(y+g)] - \overline{D}[x+f, y+g] = \\ &\quad \theta(x,y)\circ D + \theta(Dx,y) + \theta(x,Dy) + H[x,y]_{\mathfrak{a}} - Hx \circ \mathrm{ad}_{\mathfrak{a}}(y) + Hy \circ \mathrm{ad}_{\mathfrak{a}}(x). \quad (1)\end{aligned}$$

But since $\Theta = dF$ and $F(x,y) = B(Hx,y) = Hx(y)$ for all $x, y \in \mathfrak{a}$, we have

$$\begin{aligned}&\theta(x,y)Dz + \theta(Dx,y)z + \theta(x,Dy)z + H[x,y]_{\mathfrak{a}}(z) - Hx([y,z]_{\mathfrak{a}}) + Hy([x,z]_{\mathfrak{a}}) = \\ &\quad \Theta(x,y,z) + F([x,y]_{\mathfrak{a}},z) - F(x,[y,z]_{\mathfrak{a}}) + F(y,[x,z_{\mathfrak{a}}]) = \\ &\quad \Theta(x,y,z) - dF(x,y,z) = 0\end{aligned}$$

and, hence, (1) vanishes. $\square$

If the field $\mathbb{K}$ is algebraically closed, one also has the reciprocal:

Theorem 2.3 *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra over an algebraically closed field $\mathbb{K}$ which admits a skew-symmetric invertible derivation $\overline{D}$.*

There exist a Lie algebra $\mathfrak{a}$, an invertible derivation D of $\mathfrak{a}$ and a cyclic $\theta \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ such that $\mathfrak{g} = T^*_\theta \mathfrak{a}$, and the map Θ defined by $\Theta(x,y,z) = \theta(Dx,y)z + \theta(Dy,z)x + \theta(Dz,x)y$, for all$x, y, z \in \mathfrak{a}$ is a 3-coboundary for the scalar cohomology of $\mathfrak{a}$.

Proof. Let us consider the semidirect sum of Lie algebras $\mathfrak{L} = \mathrm{ad}(\mathfrak{g}) \oplus \mathbb{K}\overline{D}$. Since $\mathfrak{g}$ is nilpotent, the Lie algebra $\mathfrak{L}$ is obviously solvable. Thus, according to Lemma 3.2 in [4], we may find a maximal isotropic (with respect to the quadratic form B) ideal $\mathfrak{J}$ of $\mathfrak{g}$ which is also stable by the derivation $\overline{D}$. Now, if $\mathfrak{a} = \mathfrak{g}/\mathfrak{J}$ then $\mathfrak{a}^*$ is isomorphic to $\mathfrak{J}$ and there exists a cyclic $\theta \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ such that $(\mathfrak{g}, B)$ is isometrically isomorphic to $T^*_\theta \mathfrak{a}$ [4, Corollary 3.1]. Furhter, $\mathfrak{a}^* = \mathfrak{J}$ is stable by $\overline{D}$ and hence, there exist linear mappings $D_{11} : \mathfrak{a} \to \mathfrak{a}$, $D_{21} : \mathfrak{a} \to \mathfrak{a}^*$ and $D_{22} : \mathfrak{a}^* \to \mathfrak{a}^*$ such that $\overline{D}(x+f) = D_{11}x + D_{21}x + D_{22}f$ holds for every $x \in \mathfrak{a}$, $f \in \mathfrak{a}^*$. Clearly, D_{11} and D_{22} must be invertible since $\overline{D}$ is so. The skew-symmetry of $\overline{D}$ is equivalent to the conditions $D_{22}f = -f \circ D_{11}$, for all $f \in \mathfrak{a}^*$ and $B(D_{21}x, y) = -B(D_{21}y, x)$ for $x, y \in \mathfrak{a}$ Let $H = -D_{21}$ and $D = D_{11}$. Since $\overline{D}$ is a derivation, we get:

$$\begin{aligned}0 = [\overline{D}x, y] + [x, \overline{D}y] - \overline{D}[x,y] &= [Dx,y]_{\mathfrak{a}} + [x,Dy]_{\mathfrak{a}} - D[x,y]_{\mathfrak{a}} + \\ &H[x,y]_{\mathfrak{a}} - Hx \circ \mathrm{ad}_{\mathfrak{a}}(y) + Hy\mathrm{ad}_{\mathfrak{a}}(x) + \theta(x,y) \circ D + \theta(Dx,y) + \theta(x,Dy),\end{aligned}$$

for $x, y \in \mathfrak{a}$, which shows that D is a derivation of $\mathfrak{a}$ and that if

$$\Theta(x,y,z) = \theta(Dx,y)z + \theta(Dy,z)x + \theta(Dz,x)y$$

and F is the skew-symmetric bilinear form on $\mathfrak{a}$ defined by $F(x,y) = -B(Hx,y) = -Hx(y)$, then we have

$$\Theta(x,y,z) + F([x,y]_{\mathfrak{a}},z) - F(x,[y,z]_{\mathfrak{a}}) - F(y,[z,x]_{\mathfrak{a}}) = \Theta(x,y,z) - dF(x,y,z) = 0$$

for all $x, y, z \in \mathfrak{a}$, which finishes the proof. $\square$

Remark 3

1. It is interesting to point out that if the derivation $\overline{D}$ is semisimple then the derivation D is also semisimple. Actually, if we choose a basis of the completely isotropic ideal $\mathfrak{I}$ composed of eigenvectors of $\overline{D}$ then each element in its dual basis (with respect to B) is an eigenvector of D.

2. Theorem 2.3 does not hold in general in the case of a non-algebraically closed field. For example, an even-dimensional abelian Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ with a definite positive bilinear form is obviously a quadratic Lie algebra which admits an invertible skew-symmetric derivation. However, it cannot be a T^*-extension since there are no isotropic subspaces.

In [11] it is proved that if a Lie algebra admits an invertible derivation, then it must be nilpotent. There are however many nilpotent Lie algebras whose derivations are all singular. The following result gives a characterization of Lie algebras admitting such a derivation. Note that the result is valid for an arbitrary base field of characteristic zero (not necessarily algebraically closed). We recall that in a symplectic Lie algebra $(\mathfrak{g}, \omega)$ an ideal is called *lagrangian* if and only if it coincides with its orthogonal with respect to the form ω.

Proposition 2.4 *Let $\mathbb{K}$ be a field of characteristic zero and let $\mathfrak{a}$ be a Lie algebra over $\mathbb{K}$.*

There exists an invertible derivation of $\mathfrak{a}$ if and only if $\mathfrak{a}$ is isomorphic to the quotient Lie algebra $\mathfrak{g}/\mathfrak{I}$ of a quadratic symplectic Lie algebra $(\mathfrak{g}, B, \omega)$ by a lagrangian and completely isotropic ideal $\mathfrak{I}$.

Proof. If $\mathfrak{a}$ admits an invertible derivation then the Lie algebra $\mathfrak{g} = T_0^*\mathfrak{a}$ obtained by T^*-extension by the null cocycle $\theta = 0$ is, according to Proposition 2.2, a quadratic symplectic Lie algebra and $\mathfrak{I} = \mathfrak{a}^*$ is a lagrangian, completely isotropic ideal of $\mathfrak{g}$.

Conversely, suppose that $\mathfrak{a}$ is isomorphic to $\mathfrak{g}/\mathfrak{I}$ where $(\mathfrak{g}, B, \omega)$ is a quadratic symplectic Lie algebra and $\mathfrak{I}$ is a lagrangian completely isotropic ideal of $\mathfrak{g}$. According to [4, Corollary 3.1], $\mathfrak{g}$ is isometrically isomorphic to $T_\theta^*(\mathfrak{g}/\mathfrak{I}) = T_\theta^*\mathfrak{a}$ since $\mathfrak{I}$ is completely isotropic. Let $\overline{D} \in \mathrm{Der}_a(\mathfrak{g}, B)$ be the invertible derivation such that $\omega(X, Y) = B(\overline{D}X, Y)$ for all $X, Y \in \mathfrak{g}$. Clearly, $\omega(\mathfrak{I}, \mathfrak{I}) = \{0\}$ implies that $\overline{D}(\mathfrak{I}) \subset \mathfrak{I}^\perp = \mathfrak{I}$. Now, since, $\mathfrak{I}$ stable by $\overline{D}$, the same arguments used in the proof of Theorem 2.3 prove that the projection of $\overline{D}_{|\mathfrak{a}}$ to $\mathfrak{a}$ provides a non-singular derivation of $\mathfrak{a}$.

3 Classification of quadratic symplectic complex Lie algebras up to dimension 8

The results of the preceding section allow the classification of Lie algebras which admit a quadratic and a symplectic structure of a given dimension by the calculation of the nonisomorphic T^*-extensions of all the Lie algebras whose dimension is exactly the half. We first reduce the classification problem to the study of T^*-extensions of Lie algebras which do not admit an abelian direct summand. The main result is the following:

Proposition 3.1 *Let $\mathfrak{a} = \mathfrak{h} \oplus \mathbb{C}$ be a direct sum of Lie algebras and let $\theta \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ be a cyclic cocycle.*

(i) If the T^*-extension $\mathfrak{g} = T^*_\theta(\mathfrak{a})$ is irreducible, then there exists a Lie algebra $\mathfrak{a}_1$ and a cyclic $\theta_1 \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ such that

$$\mathfrak{g} = T^*_{\theta_1}(\mathfrak{a}_1)\,, \quad \dim \mathfrak{z}(\mathfrak{a}_1) \leq \dim \mathfrak{z}(\mathfrak{a})\,, \quad \dim\Big([\mathfrak{a}_1, \mathfrak{a}_1]_{\mathfrak{a}_1} \cap \mathfrak{z}(\mathfrak{a}_1)\Big) = \dim\Big([\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})\Big) + 1.$$

(ii) If, further, $\overline{D}$ is a semisimple skew-symmetric invertible derivation of $\mathfrak{g}$ leaving $\mathfrak{a}^*$ invariant then $\mathfrak{a}_1$ may be chosen such that $\mathfrak{a}_1^*$ is also stable by $\overline{D}$.

Proof. In order to prove (i), let $\mathcal{B} = \{x_1, x_2, \cdots, x_n, e\}$ be a basis of $\mathfrak{a}$ where $x_i \in \mathfrak{h}$, for all $i \leq n$, and $e \in \mathfrak{z}(\mathfrak{a})$ such that $e \notin \mathfrak{z}(\mathfrak{h})$ and consider its dual basis $\mathcal{B}^* = \{x_1^*, x_2^*, \cdots, x_n^*, e^*\}$. There must exist two elements $x_k, x_l \in \mathfrak{a}$ such that $\theta(x_k, x_l)(e) \neq \{0\}$ since, otherwise, $\mathfrak{g} = T^*_\theta(\mathfrak{a})$ would split as a orthogonal sum $T^*_\theta(\mathfrak{a}) = T^*_{\overline{\theta}}(\mathfrak{h}) \oplus \mathbb{C}\text{-span}\{e, e^*\}$, where $\overline{\theta}$ is the restriction of θ to $\mathfrak{h} \times \mathfrak{h}$. Recall that the center of $\mathfrak{g}$ is given by:

$$\mathfrak{z}(\mathfrak{g}) = \{x + f \in \mathfrak{a} \oplus \mathfrak{a}^*\,;\, f([y,z]_{\mathfrak{a}}) = 0, [x,y]_{\mathfrak{a}} = 0 \text{ and } \theta(x,y) = 0, \text{ for all } y, z \in \mathfrak{a}\}.$$

Therefore, $e \notin \mathfrak{z}(\mathfrak{g})$ and actually $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{a}^*$, which implies that $e^* \in \mathfrak{z}(\mathfrak{g})^\perp = [\mathfrak{g}, \mathfrak{g}]$.

Now, consider de linear subspace of $T^*_\theta(\mathfrak{a})$ given by $\mathfrak{I} = \mathbb{C}\text{-span}\{x_1^*, x_2^*, \cdots, x_n^*, e\}$. It is clear that $\mathfrak{I}$ is a subalgebra of $\mathfrak{g}$, because e is a central element of $\mathfrak{a}$, and that it is completely isotropic. Notice that $\mathfrak{I}$ actually contains $\mathfrak{h}^*$ and that $\theta(x,e)(e) = 0$ for every $x \in \mathfrak{a}$. Therefore, $\mathfrak{I}$ is an ideal of $\mathfrak{g}$ since $[e^*, x_i^*] = [e^*, e] = 0$, $[x_i, x_j^*] = x_j^* \circ \mathrm{ad}_{\mathfrak{a}}(x_i) \in \mathfrak{h}^*$ and $[x_i, e] = \theta(x_i, e) \in \mathfrak{h}^*$, for all $i, j \leq n$. Thus, according to Corollary 3.1 in [4], the algebra $\mathfrak{g}$ is isometrically isomorphic to a T^*-extension $T^*_{\theta_1}(\mathfrak{a}_1)$ of $\mathfrak{a}_1 = \mathfrak{g}/\mathfrak{I}$. Note that $\mathfrak{a}_1$ is spanned by the set $\{p(x_1), p(x_2), \cdots, p(x_n), p(e^*)\}$, where $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{I}$ denotes the canonical projection. The restriction $p_{|\mathfrak{h} \oplus \mathbb{C}e^*}$ is clearly injective and the brackets in $\mathfrak{a}_1$ are given by:

$$[p(x_i), p(x_j)]_{\mathfrak{a}_1} = p([x_i, x_j]_{\mathfrak{a}}) + \theta(x_i, x_j)(e) p(e^*)\,, [p(x_i), p(e^*)]_{\mathfrak{a}_1} = 0\,, \quad 1 \leq i, j \leq n.$$

Recall that $e^* \in [\mathfrak{g}, \mathfrak{g}]$ but it cannot be of the form $f \circ \mathrm{ad}_{\mathfrak{a}}(x)$ for $x \in \mathfrak{a}, f \in \mathfrak{a}^*$ because, otherwise, we would have $e^*(e) = f \circ \mathrm{ad}_{\mathfrak{a}}(x)(e) = 0$, an absurd. Thus, we have that $e^* \in [\mathfrak{h}, \mathfrak{h}]$ and therefore $p(e^*) \in [\mathfrak{a}_1, \mathfrak{a}_1]$. This proves that $[\mathfrak{a}_1, \mathfrak{a}_1]_{\mathfrak{a}_1} = p\Big([\mathfrak{h}, \mathfrak{h})]_{\mathfrak{a}}\Big) \oplus \mathbb{C}p(e^*)$ and further $[\mathfrak{a}_1, \mathfrak{a}_1]_{\mathfrak{a}_1} \cap \mathfrak{z}(\mathfrak{a}_1) = p\Big([\mathfrak{h}, \mathfrak{h}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{h})\Big) \oplus \mathbb{C}p(e^*)$. Since $[\mathfrak{h}, \mathfrak{h}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{h}) = [\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})$ this proves that $\dim\Big([\mathfrak{a}_1, \mathfrak{a}_1]_{\mathfrak{a}_1} \cap \mathfrak{z}(\mathfrak{a}_1)\Big) = \dim\Big([\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})\Big) + 1$. Finally, $\mathfrak{z}(\mathfrak{a}) = \mathfrak{z}(\mathfrak{h}) \oplus \mathbb{C}e$ and one has $\mathfrak{z}(\mathfrak{a}_1) = \{p(x) : x \in \mathfrak{z}(\mathfrak{h}) \text{ and } \theta(x, e) = 0\} \oplus \mathbb{C}p(e^*)$, which clearly shows that $\dim \mathfrak{z}(\mathfrak{a}_1) \leq \dim \mathfrak{z}(\mathfrak{a})$.

For the proof of (ii), if $\overline{D}$ is a semisimple invertible skew-symmetric derivation of $\mathfrak{g}$ leaving $\mathfrak{a}^*$ invariant, then, according to Theorem 2.3 and Remark 3, we have a semisimple $D \in \mathrm{Der}(\mathfrak{a})$ such that $\overline{D}(x + f) = Dx - Hx - f \circ D$ for all $x \in \mathfrak{a}$, $f \in \mathfrak{a}^*$. Since D leaves $\mathfrak{z}(\mathfrak{a})$ invariant and it is semisimple and invertible, we may choose the element $e \in \mathfrak{z}(\mathfrak{a})$ such that $e \notin \mathfrak{z}(\mathfrak{h})$ of the proof of (i) to be an eigenvector of D. Now it suffices to prove that if $\mathfrak{a}$ is constructed as above then $\mathfrak{a}_1^*$ is invariant by $\overline{D}$.

Recall that $\mathfrak{a}_1^* = \mathbb{C}\text{-span}\{x_1^*, x_2^*, \cdots, x_n^*, e\}$, where $\{x_1^*, x_2^*, \cdots, x_n^*\}$ is a basis of $\mathfrak{h}^*$. Clearly, $\overline{D}(e) \in \mathfrak{a}_1^*$ because $\overline{D}(e) = \alpha e + He$ and $0 = B(\overline{D}(e), e) = B(He, e)$ implies that the projection of He onto e^* vanishes. On the other hand, for every $j \leq n$ one has

$$B(\overline{D}(x_j), e) = -B(x_j, \overline{D}(e)) = -B(x_j, \alpha e + He) = -B(x_j, \alpha e) = 0,$$

which also shows that the projection of $\overline{D}(x_j)$ onto e^* is zero.

Corollary 3.2 *Let $\mathfrak{a}$ be a Lie algebra such that and $\theta \in Z^2(\mathfrak{a}, \mathfrak{a}^*)$ be a cyclic cocycle.*

If the T^*-extension $\mathfrak{g} = T_\theta^*(\mathfrak{a})$ is irreducible, then there exists a Lie algebra $\mathfrak{b}$ and a cyclic $\tilde{\theta} \in Z^2(\mathfrak{b}, \mathfrak{b}^*)$ such that $\mathfrak{z}(\mathfrak{b}) \subset [\mathfrak{b}, \mathfrak{b}]_{\mathfrak{b}}$ and $\mathfrak{g} = T_{\tilde{\theta}}^*(\mathfrak{b})$.

Moreover, if $\overline{D}$ is a semisimple skew-symmetric invertible derivation of $\mathfrak{g}$ leaving $\mathfrak{a}^*$ invariant then $\mathfrak{b}$ may be chosen such that $\mathfrak{b}^*$ is also stable by $\overline{D}$.

Proof. We will prove the result by induction on $l = \dim \mathfrak{z}(\mathfrak{a}) - \dim\Big([\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})\Big)$.

If $l = 0$ then obviously $\mathfrak{z}(\mathfrak{a}) \subset [\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}}$ and we can choose $\mathfrak{b} = \mathfrak{a}$. Suppose $l > 0$ and that the result is valid for all $k < l$ and consider that $l = \dim \mathfrak{z}(\mathfrak{a}) - \dim\Big([\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})\Big)$. We clearly have that $\mathfrak{a}$ is as in Proposition 3.1 and hence there exists a Lie algebra $\mathfrak{a}_1$ and a cyclic $\theta_1 \in Z^2(\mathfrak{a}_1, \mathfrak{a}_1^*)$ verifying $\mathfrak{g} = T_{\theta_1}^*(\mathfrak{a}_1)$, $\dim \mathfrak{z}(\mathfrak{a}_1) \leq \dim \mathfrak{z}(\mathfrak{a})$, and also $\dim\Big([\mathfrak{a}_1, \mathfrak{a}_1]_{\mathfrak{a}_1} \cap \mathfrak{z}(\mathfrak{a}_1)\Big) = \dim\Big([\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})\Big) + 1$. We then have

$$\dim \mathfrak{z}(\mathfrak{a}_1) - \dim\Big([\mathfrak{a}_1, \mathfrak{a}_1]_{\mathfrak{a}_1} \cap \mathfrak{z}(\mathfrak{a}_1)\Big) \leq \dim \mathfrak{z}(\mathfrak{a}) - \dim\Big([\mathfrak{a}, \mathfrak{a}]_{\mathfrak{a}} \cap \mathfrak{z}(\mathfrak{a})\Big) - 1 \leq l - 1$$

and therefore, by the induction hypothesis, there exists a Lie algebra $\mathfrak{b}$ and a cyclic $\tilde{\theta} \in Z^2(\mathfrak{b}, \mathfrak{b}^*)$ such that $\mathfrak{z}(\mathfrak{b}) \subset [\mathfrak{b}, \mathfrak{b}]_{\mathfrak{b}}$ and $\mathfrak{g} = T_{\theta_1}^*(\mathfrak{a}_1) = T_{\tilde{\theta}}^*(\mathfrak{b})$.

The second part is immediate from statement (ii) in the proposition above. $\square$

The next result gives the complete classification up to isometric isomorphism of quadratic Lie algebras with dimension less or equal that 8 admitting a symplectic structure.

First note that, up to dimension 4, a nilpotent quadratic Lie algebra must be abelian. Hence, it suffices to classify the cases of dimension 6 and 8 and therefore, according to Theorem 2.3, certain T^*-extensions of those Lie algebras of dimension 3 and 4 which admit invertible derivations. There are only 2 nonisomorphic Lie algebras of dimension 3: the abelian Lie algebra and the Heisenberg algebra $\mathcal{L}_2$, which is the algebra spanned by three elements x_1, x_2, x_3 with the only non-trivial bracket $[x_1, x_2] = x_3$. In dimension 4 one finds three nilpotent Lie algebras: the abelian 4-dimensional Lie algebra, the direct sum $\mathcal{L}_2 \oplus \mathbb{C}$ and the filiform Lie alebra $\mathcal{L}_3$, which admits a basis x_1, x_2, x_3, x_4 where the only non-trivial brackets are $[x_1, x_2] = x_3$ and $[x_1, x_3] = x_4$. According to Corollary 3.2 a quadratic symplectic Lie algebra is either reducible or a T^*-extension of a Lie algebra whose center is contained in its derived ideal. Hence, for dimension less than or equal to 8, irreducible quadratic symplectic Lie algebras are given by the non-isomorphic T^*-extensions of the filiform Lie algebras $\mathcal{L}_2$ and $\mathcal{L}_3$ by means of a 2-cocycle θ compatible with some invertible derivation (in the sense that the map Θ of Theorem 2.3 is a coboundary). The scalar cohomology and the deriviations of $\mathcal{L}_2$ and $\mathcal{L}_3$ are well-known (see, for example, [4] [8]) and one easily verifies that for $\mathcal{L}_2$ such compatibility condition leads to $\theta = 0$ whereas for $\mathcal{L}_3$, the associated 3-cocycle $\tilde{\theta} \in Z^3(\mathcal{L}_3, \mathbb{C})$ given by $\tilde{\theta}(x, y, z) = \theta(x, y)(z)$ must actually be a 3-coboundary and thus, by [4, Proposition 3.1], $T^*_\theta(\mathcal{L}_3)$ is isometrically isomorphic to the trivial T^*-extension $T^*_0(\mathcal{L}_3)$. Finally, since the only quadratic symplectic Lie algebras up to dimension 4 are the abelian Lie algebras $\mathbb{C}^2$ and $\mathbb{C}^4$, the reducible case follows at once and we get the following:

Theorem 3.3 *If $\mathfrak{g}$ is a complex quadratic Lie algebra which admits a symplectic structure and* $\dim(\mathfrak{g}) \leq 8$*, then $\mathfrak{g}$ is isometrically isomorphic to the trivial T^*-extension of one of the Lie algebras $\mathbb{C}$, $\mathbb{C}^2$, $\mathbb{C}^3$, $\mathbb{C}^4$, $\mathcal{L}_2$, $\mathcal{L}_2 \oplus \mathbb{C}$ or $\mathcal{L}_3$.*

4 Double extension of quadratic symplectic Lie algebras

In her unpublished thesis, A. Aubert [3] proved that every quadratic symplectic Lie algebra may be obtained from an abelian quadratic symplectic Lie algebra by a sequence of (generalised) symplectic double extensions by a line or a plane (see [6], [15] for the formal definition of *symplectic double extensions*). For the sake of completeness and since they will be used in Section 4, we include in this section one of Aubert's results. For our purposes concerning Manin triples of Section 5 it is more accurate to describe quadratic symplectic Lie algebras in terms of quadratic double extension instead of symplectic double extension and, hence, the statements below look slightly different from those of Aubert's. We have

considered, however, that such a difference does not justify the inclusion of the proofs in these paper.

The main tool for all the constructions of quadratic Lie algebras below is the so-called *quadratic double extension*, described as follows:

Definition 4.1 [12], [13] Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. Let $\mathfrak{b}$ another Lie algebra and $\psi : \mathfrak{b} \to \mathrm{Der}_a(\mathfrak{g}, B)$ a representation of $\mathfrak{b}$ by means of skew-symmetric derivations of $(\mathfrak{g}, B)$. Then, the map $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{b}^*$ defined by $\phi(x, y)(z) = B(\psi(z)(x), y)$, for all $x, y \in \mathfrak{g}$, $z \in \mathfrak{b}$, turns out to be a 2-cocycle for the trivial representation of $\mathfrak{g}$ in $\mathfrak{b}^*$. We now consider the central extension $\mathfrak{b}^* \times_\phi \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{b}^*$ by means of the 2-cocycle ϕ and the linear mapping $\theta : \mathfrak{b} \to \mathfrak{gl}(\mathfrak{b}^* \times_\phi \mathfrak{g})$ defined by: $\theta(x)_{|\mathfrak{b}^*} = \pi(x)$, $\theta(x)_{|\mathfrak{g}}\psi(x)$, for all $x \in \mathfrak{b}$, where π stands for the coadjoint representation of $\mathfrak{b}$. It follows that θ is a representation of $\mathfrak{b}$ in $\mathfrak{b}^* \times_\phi \mathfrak{g}$ such that $\theta(x) \in \mathrm{Der}(\mathfrak{b}^* \times_\phi \mathfrak{g})$.

The Lie algebra $\tilde{\mathfrak{g}}$ given by the semi-direct product of $\mathfrak{b}^* \times_\phi \mathfrak{g}$ by $\mathfrak{b}$ via the representation θ is called the *double extension of* $\mathfrak{g}$ *by* $\mathfrak{b}$ *by means of* ψ.

If we identify the underlying vector space of $\tilde{\mathfrak{g}}$ with the direct sum of vector spaces $\mathfrak{b} \oplus \mathfrak{g} \oplus \mathfrak{b}^*$ then the bracket in $\tilde{\mathfrak{g}}$ is given by

$$[y_1 + x_1 + f_1, y_2 + x_2 + f_2] = \Big([y_1, y_2]_{\mathfrak{b}}\Big) + \Big([x_1, x_2]_{\mathfrak{g}} + \psi(y_1)(x_2) - \psi(y_2)(x_1)\Big) + \Big(\pi(y_1)(f_2) - \pi(y_2)(f_1) + \phi(x_1, x_2)\Big),$$

for $y_1, y_2 \in \mathfrak{b}, f_1, x_1, x_2 \in \mathfrak{g}, f_2 \in \mathfrak{b}^*$. One can easily verify that the bilinear form T given by $T(y_1{+}x_1{+}f_1, y_2{+}x_2{+}f_2) = B(x_1, x_2){+}f_1(y_2){+}f_2(y_1)$, for $y_1, y_2 \in \mathfrak{b}, x_1, x_2 \in \mathfrak{g}, f_1, f_2 \in \mathfrak{b}^*$ is a scalar product on $\tilde{\mathfrak{g}}$ and hence $(\tilde{\mathfrak{g}}, B)$ becomes a quadratic Lie algebra.

In particular, if $\mathfrak{b}$ is a one-dimensional Lie algebra $\mathfrak{b} = \mathbb{K}e$, we say that $\mathfrak{g}$ is the *double extension of* $\mathfrak{a}$ *by means of the derivation* $\psi(1) = \delta$. Since this notion will be continously used from now on, we recall that in this case one may consider $\tilde{\mathfrak{g}} = \mathbb{K}e \oplus \mathfrak{g} \oplus \mathbb{K}e^*$ endowed with the bracket $[\alpha e + x + \alpha' e^*, \gamma e + y + \gamma' e^*]_{\tilde{\mathfrak{g}}}[X, Y]_{\mathfrak{g}} + \alpha\delta(y) - \gamma\delta(x) + B(\delta(x), y)e^*$, and the scalar product $T(\alpha e + x + \alpha' e^*, \gamma e + y + \gamma' e^*) = B(x, y) + \alpha\gamma' + \gamma\alpha'$, for all $x, y \in \mathfrak{g}$, $\alpha, \alpha', \gamma, \gamma' \in \mathbb{K}$.

Lemma 4.1 *Let* $\mathfrak{g}$ *be a Lie algebra,* B *an invariant scalar product on* $\mathfrak{g}$ *and* D *an invertible skew-symmetric derivation of* $\mathfrak{g}$*. Suppose that there exist* $\delta \in Der_a(\mathfrak{g}, B)$*,* $\lambda \in \mathbb{K}\backslash\{0\}$ *and* $c \in \mathfrak{g}$ *such that* $[\delta, D] - \lambda\delta = \mathrm{ad}_{\mathfrak{g}}(c)$ *and let* $(\tilde{\mathfrak{g}}, T)$ *be the quadratic double extension of* $(\mathfrak{g}, B)$ *by means of* δ*.*

The linear endomorphism $\tilde{D}$ *of* $\tilde{\mathfrak{g}}$ *defined by*

$$\tilde{D}_{|\mathfrak{g}} = D + B(c, .)e^*, \ \tilde{D}(e^*) = \lambda e^*, \ \tilde{D}(e) = -\lambda e - c,$$

is an invertible derivation of $\tilde{\mathfrak{g}}$ *which is skew-symmetric with respect to* T *.*

Remark 4 Obvously, if we set $\tilde{\omega}(X,Y) = T(\tilde{D}X,Y)$, for $X,Y \in \tilde{\mathfrak{g}}$, then $(\tilde{\mathfrak{g}},T,\tilde{\omega})$ is a quadratic symplectic Lie algebra which we will call the *quadratic symplectic double extension of* $(\mathfrak{g},B,\omega)$ *by the one-dimensional algebra by means of* (δ,c). Actually, one can easily see that $(\tilde{\mathfrak{g}},\tilde{\omega})$ is the symplectic double extension of $(\mathfrak{g},\omega)$ by means of δ and $z = D^{-1}(\delta(c))$ (see [6] ou [15] for the notion of symplectic double extension).

Theorem 4.2 *Let* $(\tilde{\mathfrak{g}},T,\tilde{\omega})$ *be a quadratic symplectic Lie algebra over a field* $\mathbb{K}$ *and let* $\tilde{D}$ *be the invertible derivation such that* $\tilde{\omega}(X,Y) = T(\tilde{D}X,Y)$ *for every* $X,Y \in \tilde{\mathfrak{g}}$.

(i) If there exists an one-dimensional central ideal of $\tilde{\mathfrak{g}}$ *which remains invariant by the derivation* $\tilde{D}$, *then* $(\tilde{\mathfrak{g}},T,\tilde{\omega})$ *is the quadratic symplectic double extension of a quadratic symplectic Lie algebra* $(\mathfrak{g},B,\omega)$ *by the one-dimensional algebra by means of a pair* (δ,c).

(ii) In particular, if $\mathbb{K}$ *is algebraically closed, then* $(\tilde{\mathfrak{g}},T,\tilde{\omega})$ *may be obtained from the two-dimensional abelian Lie algebra by a sequence of quadratic symplectic double extensions by the one-dimensional algebra*

5 Manin algebras and quadratic symplectic Lie algebras

A particular and very interesting case of quadratic double extension concerns Manin triples. Using this procedure developped in [14] and the results of section 4 we will describe more precisely the structure of quadratic symplectic Lie algebras.

Definition 5.1 Let $\mathfrak{g}$ be Lie algebra and let $\mathfrak{U}$, $\mathfrak{V}$ be two Lie subalgebras of $\mathfrak{g}$. The triple $(\mathfrak{g},\mathfrak{U},\mathfrak{V})$ is called a *Manin triple* if $\mathfrak{g}$ is the direct sum of vector subspaces $\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}$ and there exists an invariant scalar product B on $\mathfrak{g}$ such that $\mathfrak{U}$ and $\mathfrak{V}$ are completely isotropic with respect to B. In such case we will also say that $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B)$ is a *Manin algebra* and $\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}$ a *Manin decomposition* of $\mathfrak{g}$.

Remark 5

1. Obviously, any trivial T^*-extension $\mathfrak{g} = T_0^*\mathfrak{a}$ provides a Manin triple $(\mathfrak{g},\mathfrak{a}^*,\mathfrak{a})$.
2. It is well known that there is one-to-one correspondence between Lie bialgebra structures on $\mathfrak{U}$ and Manin triples $(\mathfrak{g},\mathfrak{U},\mathfrak{V})$ [5].

Theorem 5.1 [14] *Let* $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B)$ *be a Manin algebra. Let* δ *be a skew-symmetric derivation of* $(\mathfrak{G},B)$ *such that* $\delta(\mathfrak{V}) \subseteq \mathfrak{V}$.

The Lie algebra obtained by quadratic double extension of $\mathfrak{G}$ *by means of* δ *is also a Manin algebra* $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', \tilde{B})$, *where* $\mathfrak{U}'$ *is the direct sum of Lie algebras* $\mathfrak{U}' = \mathfrak{U} \oplus \mathbb{K}e^*$ *and* $\mathfrak{V}' = \mathfrak{V} \oplus \mathbb{K}e$ *with the Lie structure given by the semidirect product by means of* δ.

Definition 5.2 The Manin algebra $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', \tilde{B})$ obtained in the theorem above is called the *double extension of the Manin algebra* $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B)$ *by the one-dimensional Lie algebra (by means of* δ*).*

Proposition 5.2 *Let* $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B')$ *be a Manin algebra of dimension* n.

If either $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' \neq \{0\}$ *or* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{V}' \neq \{0\}$, *then* $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B')$ *is a double extension of a Manin algebra* $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B)$ *of dimension* $n-2$ *by the one-dimensional Lie algebra.*

Proof. If $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' \neq \{0\}$, then there exists $e^* \in \mathfrak{z}(\mathfrak{g}) \backslash \{0\}$ such that $\mathfrak{I} = \mathbb{K}e^* \subseteq \mathfrak{U}'$ and, hence, there exists $e \in \mathfrak{V}'$ such that $B'(e^*, e) = 1$. Consequently, if $\mathfrak{G} = (\mathbb{K}e^* \oplus \mathbb{K}e)^{\perp}$ denotes the orthogonal of $\mathbb{K}e^* \oplus \mathbb{K}e$ with respect to B', one easily verifies that $B := B'_{|\mathfrak{G}\times\mathfrak{G}}$ is an invariant scalar product on $\mathfrak{G}$ and that $(\mathfrak{g}, B')$ is the quadratic double extension of $(\mathfrak{G}, B)$ by means of the derivation $\delta = \mathrm{ad}_{\mathfrak{g}}(e)_{|\mathfrak{G}}$.

Now, $\mathfrak{U} = \mathfrak{G} \cap \mathfrak{U}'$ is a Lie subalgebra of $\mathfrak{G}$ and $\mathfrak{U}' = \mathbb{K}e^* \oplus \mathfrak{U}$ because $\mathfrak{U}'$ is a subalgebra of $\mathfrak{g}$ and $e^* \in \mathfrak{U}'$. Moreover, $B(\mathfrak{U}, \mathfrak{U}) = B'(\mathfrak{U}, \mathfrak{U}) = \{0\}$. The fact that $\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}'$ and $\mathfrak{U}' \subseteq \mathfrak{I}^{\perp}$ implies that $\mathfrak{I}^{\perp} = \mathfrak{U}' \oplus (\mathfrak{V}' \cap \mathfrak{I}^{\perp})$. Since $B'(\mathfrak{V}' \cap \mathfrak{I}^{\perp}, \mathbb{K}e^* \oplus \mathbb{K}e) = \{0\}$, we immediately get that $\mathfrak{V}' \cap \mathfrak{I}^{\perp} \subseteq \mathfrak{G}$ and, furhter, $\mathfrak{G} = (\mathfrak{U}' \cap \mathfrak{G}) \oplus (\mathfrak{V}' \cap \mathfrak{I}^{\perp})$ because $\mathfrak{G} \subseteq \mathfrak{I}^{\perp}$. Consequently, $\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}$ where $\mathfrak{V} = \mathfrak{V}' \cap \mathfrak{I}^{\perp}$ is a Lie subalgebra of $\mathfrak{G}$ such that $B(\mathfrak{V}, \mathfrak{V}) = \{0\}$. This proves that $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B)$ is a Manin algebra of $\dim(\mathfrak{G}) = \dim(\mathfrak{g}) - 2$. Finally, since $e \in \mathfrak{V}'$, then $\delta(\mathfrak{V}) = \mathrm{ad}_{\mathfrak{g}}(e)(\mathfrak{V}) \subseteq \mathfrak{V}$ and we conclude that $\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}'$ is the double extension of the Manin algebra $\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}$ by means of δ. $\square$

Theorem 5.3 *Let* $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B')$ *be a non-zero Manin algebra. If* $\mathfrak{g}$ *is a nilpotent Lie algebra, then* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' \not\{0\}$ *and* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{V}' \neq \{0\}$. *As a consequence,* $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B')$ *is a double extension of a Manin algebra of dimension* $\dim(\mathfrak{g}) - 2$.

Proof. Since $\mathfrak{g}$ and $\mathfrak{U}'$ are non-zero nilpotent Lie algebras then $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$ and $\mathfrak{z}(\mathfrak{U}') \neq \{0\}$. Suppose that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' = \{0\}$. We set $\mathfrak{L}_0 = [\mathfrak{V}', \mathfrak{z}(\mathfrak{U}')]$, it is clear that $\mathfrak{L}_0 \neq \{0\}$ and $[\mathfrak{U}', \mathfrak{L}_0] \subseteq \mathfrak{L}_0$. Let us consider $v \in \mathfrak{V}'$ and $u \in \mathfrak{z}(\mathfrak{U}')$ and write $[v, u] = x + y$ where $x \in \mathfrak{U}'$ and $y \in \mathfrak{V}'$. Now, if $t \in \mathfrak{U}'$, then $B(y, t) = B'([v, u], t) = B'(v, [u, t]) = 0$ and the non-denereracy of B' shows that $y = 0$. It follows that $\mathfrak{L}_0 \subseteq \mathfrak{U}'$. Therefore, $\mathfrak{L}_0$ is a non-zero ideal of $\mathfrak{U}'$, which implies that $\mathfrak{L}_0 \cap \mathfrak{z}(\mathfrak{U}') \neq \{0\}$ because $\mathfrak{U}'$ is a nilpotent Lie algebra.

Now, consider $\mathfrak{L}_1 = [\mathfrak{V}', \mathfrak{L}_0 \cap \mathfrak{z}(\mathfrak{U}')]$. The fact that $\mathfrak{L}_0 \cap \mathfrak{z}(\mathfrak{U}') \neq \{0\}$ and $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' = \{0\}$ implies that $\mathfrak{L}_1 \neq \{0\}$. Further, as B' is an invariant scalar product on $\mathfrak{g}$, we have that $\mathfrak{L}_1 \subseteq \mathfrak{U}'$. Since $[\mathfrak{U}', \mathfrak{L}_1] = [\mathfrak{L}_0 \cap \mathfrak{z}(\mathfrak{U}'), [\mathfrak{U}', \mathfrak{V}']] \subseteq [\mathfrak{L}_0 \cap \mathfrak{z}(\mathfrak{U}'), \mathfrak{V}'] = \mathfrak{L}_1$, then $\mathfrak{L}_1$ is an ideal of $\mathfrak{U}'$. Consequently, $\mathfrak{L}_1 \cap \mathfrak{z}(\mathfrak{U}') \neq \{0\}$. If we repeat this process sucessively, we get a sequence $(\mathfrak{L}_n)_{n\in\mathbb{N}}$ of non-zero ideals of $\mathfrak{U}'$ defined by $\mathfrak{L}_0 = [\mathfrak{V}', \mathfrak{z}(\mathfrak{U}')]$ and $\mathfrak{L}_n = [\mathfrak{V}', \mathfrak{L}_{n-1} \cap \mathfrak{z}(\mathfrak{U}')]$, for $n \geq 1$. Further, it is obviously verified that $\mathfrak{L}_n \subseteq \mathcal{C}^{n+1}(\mathfrak{g})$, where $(\mathcal{C}^n(\mathfrak{g}))_{n\in\mathbb{N}}$ stands for the central descending series of $\mathfrak{g}$. The fact that $\mathfrak{g} \neq \{0\}$ is a nilpotent Lie algebra

implies that there exists $k \in \mathbb{N}$ such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$. Consequently, $\mathfrak{L}_{k-1} = \{0\}$ which contradicts the fact that $\mathfrak{L}_n \neq \{0\}$ for all $n \in \mathbb{N}$. We conclude that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' \neq \{0\}$. The same reasoning shows that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{V}' \neq \{0\}$. Now, the final part of the statement is immediate from Proposition 5.2. $\square$

Corollary 5.4 *Every nilpotent Manin algebra can be obtained by a sequence of quadratic double extensions by one-dimensional Lie algebras starting from a 2-dimensional abelian Manin algebra.*

Definition 5.3 We will say that a Manin algebra $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B)$ is a *special symplectic Manin algebra* if there exists a symplectic structure ω on the Lie algebra $\mathfrak{g}$ such that $\omega(\mathfrak{U}, \mathfrak{U}) = \omega(\mathfrak{V}, \mathfrak{V}) = \{0\}$.

The following lemma follows easily:

Lemma 5.5 *A Manin algebra $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B)$ is special symplectic if and only if there exists an invertible skew-symmetric derivation D of $(\mathfrak{g}, B)$ such that $D(\mathfrak{U}) \subseteq \mathfrak{U}$ and $D(\mathfrak{V}) \subseteq \mathfrak{V}$.*

Notation If a Manin algebra $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B)$ admits a special symplectic structure we will note the corresponding symplectic quadratic algebra by either $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B, \omega)$ or by $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B, D)$, where D denotes the derivation given in the lemma.

Our main interest in the study of special symplectic Manin algebras lies in the fact that every quadratic symplectic Lie algebra is so. More precisely:

Proposition 5.6 *Let $(\mathfrak{g}, B, \omega)$ be a quadratic symplectic Lie algebra over an algebraically closed field $\mathbb{K}$. There exist two Lie subalgebras $\mathfrak{U}$ and $\mathfrak{V}$ of $\mathfrak{g}$ such that $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B, \omega)$ is a special symplectic Manin algebra.*

Proof. Consider $D \in \mathrm{Der}_a(\mathfrak{g})$ invertible such that $\omega(x, y) = B(D(x), y)$ for all $x, y \in \mathfrak{g}$. The proof follows the same argument used in [1, Corollary 2.16]. Let $\mathrm{Sp}(D)$ denote the set of all distinct eigenvalues of D. Since every field of characteristic zero is a $\mathbb{Q}$-vector space, we can consider the $\mathbb{Q}$-linear span $\mathcal{S}$ of $\mathrm{Sp}(D)$ (notice that $\dim_{\mathbb{Q}} \mathcal{S} \leq \dim_{\mathbb{K}} \mathfrak{g}/2$ since D is skew-symmetric). Let us fix a basis $\mathcal{H} = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ of $\mathcal{S}$ and let us consider on $\mathcal{S}$ the lexicographic order with respect to $\mathcal{H}$, this is to say: $\sum_{i=1}^k m_i \alpha_i > 0$ if and only if there exists $i_0 \leq k$ such that $m_{i_0} > 0$ and $m_j = 0$ for $j < i_0$. Obviouosly, with this order we have that $\alpha + \beta > 0$ if $\alpha, \beta > 0$. Now, define Sp^+ and Sp^- respectively as the sets of positive and negative elements of $\mathrm{Sp}(D)$. It is clear that $\mathrm{Sp}(D) = \mathrm{Sp}^+ \cup \mathrm{Sp}^-$ and $\mathrm{Sp}^+ \cap \mathrm{Sp}^- = \emptyset$. For each $\lambda \in \mathrm{Sp}(D)$, let us consider $\mathfrak{g}(\lambda) = \{x \in \mathfrak{g} \,:\, (D - \lambda \mathrm{id}_{\mathfrak{g}})^{\dim(\mathfrak{g})}(x) = 0\}$. The fact that D is skew-symmetric with respect to B implies that $B(\mathfrak{g}(\lambda), \mathfrak{g}(\lambda')) = \{0\}$ for all $\lambda, \lambda' \in \mathrm{Sp}(D)$ such that $\lambda + \lambda' \neq 0$. Now, if we set

$$\mathfrak{U} = \sum_{\lambda \in \mathrm{Sp}^+} \mathfrak{g}(\lambda)\,, \quad \mathfrak{V} = \sum_{\lambda \in \mathrm{Sp}^-} \mathfrak{g}(\lambda),$$

then $\mathfrak{U}$ and $\mathfrak{V}$ are two completely isotropic Lie subalgebras of $\mathfrak{g}$ which are stable by D. We conclude that $(\mathfrak{g} = \mathfrak{U} \oplus \mathfrak{V}, B, \omega)$ is a special symplectic Manin algebra. $\square$

The inductive description of special symplectic Manin algebras may also be done by a double extension procedure using only Manin algebras. In order to show this, we begin with the following analog of Lemma 4.1:

Lemma 5.7 *Let $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B, D)$ be a special symplectic Manin algebra and suppose that there exist $\delta \in \mathrm{Der}_a(\mathfrak{G}, B)$, $\lambda \in \mathbb{K}\backslash\{0\}$ and $c \in \mathfrak{V}$ such that $[\delta, D] - \lambda\delta = \mathrm{ad}_{\mathfrak{G}}(c)$ and $\delta(\mathfrak{V}) \subseteq \mathfrak{V}$.*

Let $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B')$ be the double extension of the Manin algebra $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B)$ by means the derivation δ (constructed as in Theorem 5.1).

The skew-symmetric invertible derivation $\tilde{D}$ of $(\mathfrak{g}, B')$ defined by:

$$\tilde{D}_{|_{\mathfrak{G}}} = D + B(c, .)e^*, \;\; \tilde{D}(e^*) = \lambda e^*, \;\; \tilde{D}(e) = -\lambda e - c,$$

verifies $\tilde{D}(\mathfrak{U}') \subseteq \mathfrak{U}'$ and $\tilde{D}(\mathfrak{V}') \subseteq \mathfrak{V}'$. Consequently, $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B', \tilde{D})$ is a also a special symplectic Manin algebra.

Proof. Bearing in mind Lemma 4.1, it suffices to prove that $\mathfrak{U}$ and $\mathfrak{V}$ are stable by $\tilde{D}$. Since $D(\mathfrak{U}) \subseteq \mathfrak{U}$ and $\tilde{D}(e^*)\lambda e^*$, we immediately get $\tilde{D}(\mathfrak{U}') \subseteq \mathfrak{U}'$. The fact that $c \in \mathfrak{V}$ and $B(\mathfrak{V}, \mathfrak{V}) = \{0\}$ implies that $\tilde{D}(\mathfrak{V}) = D(\mathfrak{V}) \subseteq \mathfrak{V}$. Consequently $\tilde{D}(\mathfrak{V}') \subseteq \mathfrak{V}'$ because $\tilde{D}(e) = -\lambda e - c$. $\square$

The special symplectic Manin algebra $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B', \tilde{D})$ of the Lemma will be called the *special double extension of the symplectic Manin algebra* $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B, D)$ *by means of* (δ, c).

Theorem 5.8 *Let $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B', \tilde{D})$ be a special symplectic Manin algebra over $\mathbb{K}$.*

(i) If $\tilde{D}$ admits an eigenvector $z \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' + \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{V}'$, then $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B', \tilde{D})$ is a special double extension of a special symplectic Manin algebra $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B, D)$ by means of a pair (δ, c) where $c \in \mathfrak{V}$ and δ is a skew-symmetric derivation of $\mathfrak{G}$ leaving $\mathfrak{V}$ invariant.

(ii) In particular, if $\mathbb{K}$ is algebraically closed, then $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B', \tilde{D})$ may be obtained from a two-dimensional special symplectic Manin algebra by a sequence of special double extensions by the one-dimensional algebra.

Proof. Let us consider $z \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}' + \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{V}'$, $z \neq 0$ such that $\tilde{D}z = \lambda z$ for some $\lambda \in \mathbb{K}\backslash\{0\}$ and put $z = u + v$ where $u \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}'$, $v \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{V}'$. Since both U' and V' are stable by $\tilde{D}$, it is clear that $\tilde{D}u = \lambda u$ and $\tilde{D}v = \lambda v$ and either $u \neq 0$ or $v \neq 0$. We will

suppose that $u \neq 0$ (otherwise, we can change the roles of U' and V'). If we set $e^* = u$ then we have, as in Proposition 5.2 that $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B')$ is the double extension of a Manin algebra $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B)$ where $\mathfrak{U}' = \mathfrak{U} \oplus \mathbb{K}e^*$, $\mathfrak{V}' = \mathfrak{V} \oplus \mathbb{K}e$ and $B'(e^*, e) = 1$ by means of the derivation $\delta = \mathrm{ad}_{\mathfrak{g}}(e)_{|\mathfrak{G}}$.

Since the ideal $\mathfrak{I} = \mathbb{K}e^*$ is invariant by the skew-symmetric mapping $\tilde{D}$, so is its orthogonal $\mathfrak{I}^{\perp} = \mathbb{K}e^* \oplus \mathfrak{G}$. Now, if $p : \mathfrak{I}^{\perp} = \mathbb{K}e^* \oplus \mathfrak{G} \to \mathfrak{G}$ denotes the projection $p(\alpha e^* + x) = x$ for $\alpha \in \mathbb{K}, x \in \mathfrak{G}$, then one can easily verify that $D = p \circ \tilde{D}_{|\mathfrak{G}}$ is an invertible skew-symmetric derivation of $(\mathfrak{G}, B)$. Further, as $\mathfrak{U} = \mathfrak{U}' \cap \mathfrak{G}$ and $\mathfrak{U}'$ is invariant by $\tilde{D}$, we have that $D(\mathfrak{U}) \subset \mathfrak{U}$ and, analogously, we prove $D(\mathfrak{V}) \subset \mathfrak{V}$. Since $\tilde{D}$ is skew-symmetric, $B(e^*, e) = 1$ and $\tilde{D}(e) \in \mathfrak{V}'$, one immediately obtains that there exists $c \in V$ such that $\tilde{D}(e) = -\lambda e - c$ and $\tilde{D}_{|\mathfrak{G}} = D + B(c, .)e^*$. Further, from the fact that $\tilde{D}$ is a derivation, one easily deduces that $[\delta, D'] - \lambda\delta = \mathrm{ad}_{\mathfrak{G}}(c)$ and, therefore, $(\mathfrak{g} = \mathfrak{U}' \oplus \mathfrak{V}', B', \tilde{D})$ is the special double extension of $(\mathfrak{G} = \mathfrak{U} \oplus \mathfrak{V}, B, D)$ by means of (δ, c) as claimed in (i).

Now, if $\mathbb{K}$ is algebraically closed, we can always find an eigenvector of $\tilde{D}$ in $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{U}'$ since, according to Theorem 5.3, it is a non-zero subspace and it is obviously stable by all derivations of $\mathfrak{g}$ which leave $\mathfrak{U}'$ invariant. If we apply (i) sucessively then (ii) follows. □

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